The Lee Fields Medal V

Time Allowed: Up to Two Hours and 15 Minutes

Tables and calculators may be used.

1. Find three integers in arithmetic sequence whose product is a prime.

2. Consider a unit circle on the plane of radius $r = 1$ and centre $O(0, 0)$ and a rotation angle $\theta \in [0, 2\pi]$. Starting with the point $x_0 = (1, 0)$, rotate x_0 through the angle θ to get a new point x_1. Then rotate x_1 through θ to get a new point x_2. Then rotate x_2 through θ to get x_3, etc. For example, for $\theta = \pi/2 = 90°$:

$$\begin{aligned} x_0 &= (1, 0) \\ x_1 &= (0, 1) \\ x_2 &= (-1, 0) \\ x_3 &= (0, -1) \\ x_4 &= (1, 0), \end{aligned}$$

   and the sequence of points repeats itself in this case.

   Show that no matter what rotation angle θ you pick, for large enough N, the sequence

$$x_0,\ x_1,\ x_2,\ x_3, \ldots, x_N$$

   contains two points such that the distance between the two points is less than $1/100$.

3. Suppose that i is a number such that $i^2 = -1$. Suppose further that

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots, \\ \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \text{ and} \\ \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \end{aligned}$$

   Show that

$$e^{it} = \cos(t) + i\sin(t),$$

   and that

$$e^{i\pi} + 1 = 0.$$

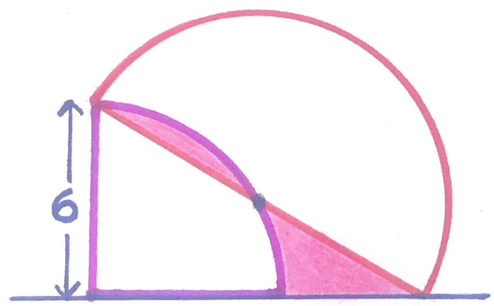
4. Let r, $m > 0$ be real numbers

   (a) Produce a rough sketch of the curves $y = mx$ and $x^2 + y^2 = r^2$.

   (b) Using this rough sketch, write down how many solutions the below set of simultaneous equations has:

$$y = mx$$
$$x^2 + y^2 = r^2$$

5. The centre of the semicircle lies on the perimeter of the quarter circle. What's the shaded area?



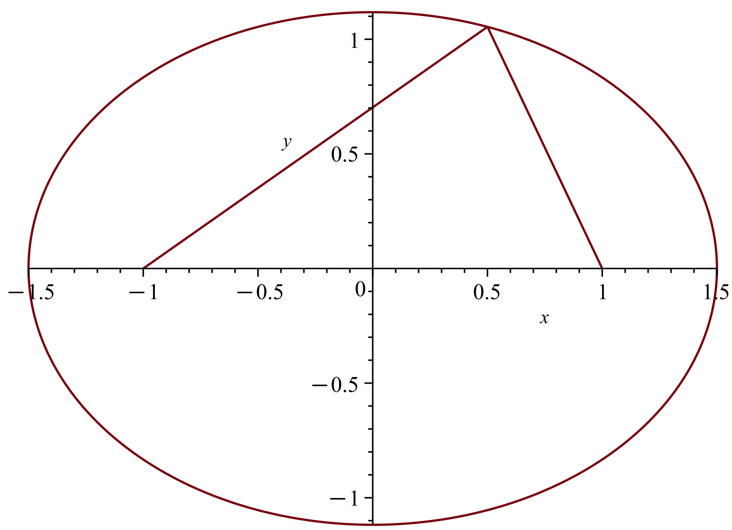
6. Prove that for all $0° < \theta < 90°$,

$$\tan\theta > \sin\theta.$$

7. Construct a finite list of data/numbers with a unique mode such that

$$\text{median} < \text{mean} < \text{mode}\,.$$

8. An ellipse is a curve surrounding two focal points F_1, and F_2, such that for all points P on the curve, the sum of the distances $|F_1P|$ and $|PF_2|$ is a constant. In the ellipse below, the foci are $(-1, 0)$ and $(1, 0)$, and the constant is four.



Write down the/an equation of the ellipse.

9. A grandfather has one son and one daughter. The grandfather asks both his son and his daughter: "How many children do you have?". They answer, and he finds the mean-average of their answers. Then the grandfather asks all his grandchildren: "One of your parents is my child. How many children does that parent have?". They answer, and he finds the mean-average of their answers.

   (a) Show that the second answer is always greater than or equal to the first answer.

   (b) Under what conditions is the first answer equal to the second answer?

10. Consider an A4 (210 mm $\times$ 297 mm) cardboard sheet. Suppose the cardboard is 1 mm thick. Prove that the sheet cannot be folded nine times.